Presence of chaos in a self-organized critical system

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We investigate a one-dimensional chain of blocks and springs driven on a surface with friction. We find that, as the number of blocks N increases, the system has a power law dependence on the size of slipping events, characteristic of self-organized criticality. The largest Liapunov exponent also increases with increasing N, approaching an asymptotic value at large N. Thus, contrary to a previous conjecture [P. Bak and C. Tang, J. Geophys. Res. B **94**, 15 635 (1989)], strong chaos (positive Liapunov exponent) and self-organized criticality can coexist.

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Dissipative driven systems with many degrees of freedom can naturally reach a critical state characterized by power law distributions of avalanchelike events. The power laws are limited only by the size of the system and of its elementary cell. This phenomenon was called self-organized criticality (SOC) by Bak, Tang, and Wiesenfeld [1]. It has been observed in several systems, such as models for sandpiles [1], models for earthquakes [2], and charge density waves [3].

It was reported by Bak, Tang, and Chen that, in the self-organized critical systems that they examined, the divergence of nearby trajectories followed a power law [2]. This implies that the largest Liapunov exponent (LLE) of the system is zero. The Liapunov exponent is a quantity widely used in the study of dynamical systems [4]. A system is generally considered to be chaotic if the LLE is greater than zero, in which the divergence of nearby trajectories occurs exponentially. Bak, Tang, and Chen called power law divergence of trajectories "weak chaos," and conjectured that this was a universal property of SOC systems [2]. SOC systems in which the LLE was found to be zero were cellular automaton models for earthquakes [2], cellular automaton model for sandpiles [5], and, more recently, a coupled map lattice [6].

In this paper we show that SOC does not necessarily imply a zero LLE. We study a mechanical system governed by coupled ordinary differential equations and show that chaos and a nonzero LLE can coexist. The system we study was introduced in 1967 by Burridge and Knopoff [7] as a mechanical model for the stick-slip behavior observed in earthquake faults. It consists of blocks connected by springs and driven with constant velocity on a surface with friction. Only the first element is connected to the driving mechanism. This system has been called the "train model" [8]. Burridge and Knopoff studied experimentally a small chain of eight blocks of this model and showed that its dynamics is characterized by the presence of sudden displacements of a group of blocks (avalanches or quakes), which come to rest after a certain time. This is what is called stick-slip dynamics. They observed a power law behavior for the distribution (frequency) of the potential energy released during the avalanches. A numerical study of the avalanche sizes in much larger chains was performed in [8]. It was shown that the train model presents SOC. Recently, an analysis of the nonlinear dynamics of a system of two blocks in the train model was done [9]. It was found that this small system presents a very rich dynamics, with periodic, quasiperiodic, and chaotic orbits. Several routes to chaos were identified, such as period-doubling bifurcations [10] and two types of intermittencies [11]. This system is therefore a good test bed to examine the LLE as the system size increases. Another spring-block system introduced by Burridge and Knopoff has received considerable attention recently [12,13], with its chaotic dynamics being studied in [14–16]. In this other model SOC *is not observed*, since the power law distributions have a limited extent [12,13], i.e., a correlation length is observed in it, which is smaller than the size of the chain. Therefore, we do not examine the Liapunov exponent of this alternative model.

In this paper we study the train model over a range of system sizes, from the chaotic behavior observed in small systems to the SOC dynamics found in large systems, to show how the LLE evolves as the size of the chain increases. Each block of the system has mass m and the springs have an elastic constant k. The first block is pulled with constant velocity v and the friction force F between each block and the surface is a function of the instantaneous velocity of the block with reference to a characteristic velocity v_c (see Fig. 1 of Ref. [8]). The equations of motion for the blocks are given by

$$\ddot{X}_j = k(X_{j+1} - 2X_j + X_{j-1}) - F(\dot{X}_j/v_c), \qquad (1)$$

where X_j denotes the displacement of the block measured with respect to the position where the sum of the elastic forces on it is zero. These equations are applicable only when the respective block is moving and the sum of the elastic forces in the block is larger than the maximum

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force of static friction. If these conditions are not met we simply have $\dot{X}_j = 0$. Here we consider open boundary conditions, with $X_0 = vt$ and $X_{N+1} = X_N$. If we write the friction force as $F(\dot{X}_j/v_c) = F_0\Phi(\dot{X}_j/v_c)$, where $\Phi(0) = 1$, and introduce the variables $\tau = \omega_p t$, $\omega_p^2 = k/m$, and $U_j = kX_j/F_0$, Eq. (1) can be written in the dimensionless form [8]

$$\ddot{U}_j = U_{j+1} - 2U_j + U_{j-1} - \Phi(\dot{U}_j/\nu_c), \tag{2}$$

with $\nu = v/V_0$, $\nu_c = v_c/V_0$, and $V_0 = F_0/\sqrt{km}$. Overdots now denote differentiation with respect to τ. In a system of a single block the quantity F_0/ω_p is the maximum displacement of the pulling spring before the block starts to move; in the absence of dynamical friction $2\pi/\omega_p$ and V_0 are, respectively, a characteristic period of oscillation of the block and the maximum velocity it attains. The boundary conditions are $U_0 = \nu\tau$ and $U_{N+1} = U_N$. This system is therefore described by two dimensionless parameters ν and ν_c. The system is $2N$ dimensional since its evolution is completely specified by giving the initial positions and velocities of the blocks. We use the velocity-weakening friction force given by [12]

$$\Phi(\dot{U}/\nu_c) = \frac{\mathrm{sgn}(\dot{U})}{1 + |\dot{U}|/\nu_c}, \tag{3}$$

which is a simple nonlinear function. The friction force is the only nonlinear element in this model.

The stick-slip motion observed in the system occurs in the following way. Suppose that at the initial instant all the blocks are at rest. As the time evolves the first spring is stretched by the driving mechanism until the force applied to the first mass exceeds the static frictional force, at which time the first mass moves. It slips a certain distance and stops. This reduces the extension in the first spring, but at the same time stretches the second spring. The one-block avalanches continue to occur until the spring force on the second block exceeds the static frictional force. Then an avalanche involving two blocks is observed and the spring that connects the second to the third block is stretched. Thus avalanches involving three, four, and more blocks appear during the time evolution. Finally, we see a larger avalanche involving all the blocks of the chain, which rebuilds the system. A new sequence of avalanches starts. Note that in this model an avalanche that involves the ith element of the chain necessarily involves all the blocks with $j < i$.

We have done numerical studies in a wide range of parameter values of the frequency of the avalanches as a function of the number of blocks involved in it and as a function of the displacement of the blocks. We have found power law distributions in both cases [8] and our numerical studies also show power law behavior for the distribution of the duration of the avalanches. In this way, we show that the train model displays SOC. A typical solution of the frequency of the avalanches $\rho(M > M')$ in which the displacement M of the blocks is greater than M' is shown in Fig. 1. In the numerical simulations of this paper we have generally started the system with the blocks at rest and with the sum of the elastic forces in each block equal to zero. Other initial conditions were considered and the results did not change. Before we start to compile statistics we let the system evolve until it reaches a statistical stationary state.

For a small system we can use the standard techniques of chaos theory to study the change of the dynamics when the nonlinearity increases, i.e., increasing ν_c^{-1}, and fixing the pulling velocity. This is usually done by plotting bifurcation diagrams. For a two-block system, a Poincaré section [4] lowers its dimensionality from 4 to 3, for which the attractors are still difficult to visualize. We therefore display the motion of the center of mass, which we have verified gives the same topology of the dynamics of the two-block system. However, we do not expect that this approach would always be good for a system with a large number of blocks. We have found [9] that for $N = 2$ the dynamical evolution of the block j occurs around a point, here denoted by U_j^e, that corresponds to its coordinate in the unstable solution in which the system moves with constant velocity equal to the pulling velocity. The coordinate of the center of mass with respect to U_j^e is

$$W = \frac{U_1 - U_1^e + U_2 - U_2^e}{2}, \tag{4}$$

with $U_1^e = -2/(1+\nu/\nu_c)+\nu\tau$ and $U_2^e = -3/(1+\nu/\nu_c) + \nu\tau$. We take the Poincaré section of $\dot{W}$ at $W = 0$. In this way, we reduce the dynamics to the study of a single variable. We show in Fig. 2 a bifurcation diagram for the pulling velocity $\nu = 0.1$. On the x axis we have ν_c^{-1} and on the y axis we plot $\dot{W}$ at $W = 0$. In the diagrams we can see windows of periodic motion and regions where the motion is nonperiodic. We find regions with period doubling bifurcation route to chaos [10], as well as intermittencies of types I and II [11].

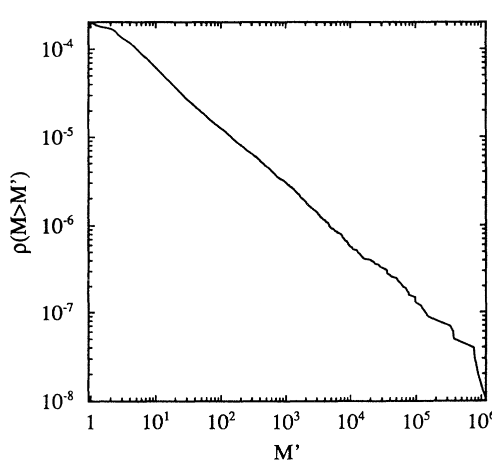

FIG. 1. Frequency of the avalanches $\rho(M > M')$ in which the displacement of the blocks M is greater than M'. The parameters are $\nu_c^{-1} = 0.8$, $\nu = 0.1$, and $N = 200$. The number of avalanches is 30 000 and $\rho(M > M')$ was divided by the number of blocks in the chain.

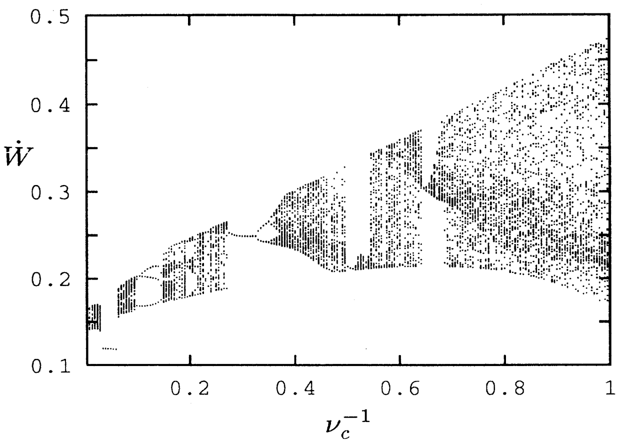

FIG. 2. Bifurcation diagram of the velocity of center of mass $\dot{W}$ on the surface of section $W = 0$ as a function of ν_c^{-1} for $\nu = 0.1$ and $N = 2$.

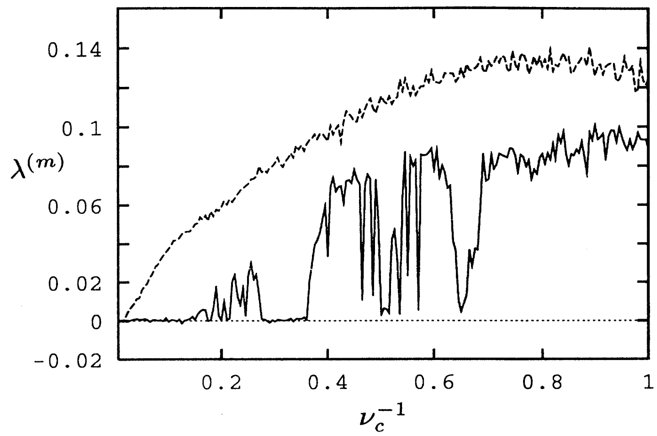

FIG. 3. The largest Liapunov exponent as a function of ν_c^{-1} with $\nu = 0.1$ for $N = 2$ (solid line), which corresponds to the bifurcation diagram shown in Fig. 2, and $N = 10$ (dashed line). The calculation of $\lambda^{(m)}$ is done for an integration time $\tau = 30\,000N$, with time steps of $\Delta\tau = 0.01$ and perturbations to the position and velocities of each block equal to 10^{-5}.

To calculate the LLE we use the standard algorithm introduced in [17]. The method is the following. Denote by $\mathbf{U}(t)$ the $2N$-dimensional vector that gives the trajectory of the attractor in phase space. Then, at a given instant t_0 take a point $\mathbf{W}(t_0) = \mathbf{U}(t_0) + \mathbf{\Delta}(t_0)$, where $\mathbf{\Delta}(t_0)$ is an arbitrary vector with a small norm. Next, evolve both $\mathbf{W}(t_0)$ and $\mathbf{U}(t_0)$. Then, calculate the separation $\mathbf{\Delta}(t_1) = \mathbf{W}(t_1) - \mathbf{U}(t_1)$ of the orbits after a small time interval $\delta t = t_1 - t_0$. Since the LLE is a function of the average separation of *nearby* trajectories along the attractor, it is necessary to renormalize $\mathbf{\Delta}(t_1)$ to form the vector $\mathbf{W}'(t_1) = \mathbf{U}(t_1) + \epsilon\mathbf{\Delta}(t_1)/|\mathbf{\Delta}(t_1)|$, where $\epsilon \ll 1$ and $|\,|$ denotes the norm. Now evolve $\mathbf{W}'(t_1)$ and $\mathbf{U}(t_1)$ and calculate the new separation of the two orbits $\mathbf{\Delta}(t_2)$. The process is reapeated p times. The LLE is given by

$$\lambda^{(m)} = \frac{1}{p\delta t}\sum_{i=1,p}\log_2\left[\frac{|\mathbf{\Delta}(t_i)|}{\epsilon}\right], \qquad (5)$$

which will converge for large p.

In Fig. 3 we show (solid line) $\lambda^{(m)}$ for the bifurcation diagram displayed in Fig. 2. We see regions in which the motion is nonperiodic and $\lambda^{(m)} = 0$. This corresponds to the quasiperiodic motion observed for small ν_c^{-1}. The entrance into chaos ($\lambda^{(m)} > 0$) for this system occurs around $\nu_c^{-1} = 0.146$. Several windows of periodic motion are found, which give $\lambda^{(m)} = 0$, as expected in flows. Our studies have shown that the LLE is not very sensitive to the pulling velocity if ν is not large and ν_c^{-1} not very small [9].

The investigation of the LLE in larger systems show that if periodic motion exits at all, they are limited to very narrow windows. We have not been able to see any periodic window for systems with $N > 2$. We found quasiperiodicity when ν_c^{-1} is small; at larger nonlinearity (larger ν_c^{-1}) only chaotic motion was found. For a fixed pulling velocity, the value of ν_c^{-1}, where the first entrance into chaos occurs, decreases as N increases. As an example, we show in Fig. 3 (dashed line) the LLE for a system with ten blocks, where the entrance into chaos occurs around $\nu_c^{-1} = 0.017$. Thus, for increasingly large systems the chaotic behavior fills an increasingly large part of the parameter space.

In Fig. 4 we show how the LLE varies with an increasing number of blocks. We plot three cases for different parameter values. In all cases, we see that for small chains the Liapunov exponent tends to increase with N, with some fluctuations. As the chain gets larger we find that within the statistical error, the LLE exponent asymptotes to a constant value. For large N the error bars (not shown) are smaller than the symbol size. Since for large chains the presence of SOC is found (see Fig. 1), we conclude that SOC is not always characterized by a zero LLE.

The LLE can also be calculated by the expression $\lambda^{(j)} = \sum_{i=1,p}\log_2(|\omega_i^{(j)}|)/p$, where ω_i are the eigenvalues of the product of the Jacobian matrices for a given orbit. For aperiodic orbits, as well as for periodic ones, the Liapunov exponents will converge when the limit $p \to \infty$ is taken. For numerical calculations the definition given above is impractical because the product of Jacobian matrices will exhibit underflow or overflow problems as p increases. This problem can be avoided by using a method introduced by Sano and Sawada [18], where after each integration step a Gram-Schmidt orthonormalization procedure is used to avoid the underflow or overflow of the evolved vectors. An analytical calculation of the LLE using this method becomes increasingly cumbersome for an arbitrary orbit as p increases. However, we can obtain an analytical expression for the LLE at the fixed point for which all the blocks move with constant velocity, equal to the pulling velocity. If we linearize the friction force around the pulling velocity, using

$$A \equiv d\Phi/d\dot{U}|_{\dot{U}=\nu} = -\nu_c/(\nu_c+\nu)^2, \qquad (6)$$

we have for the linearized equation of motion

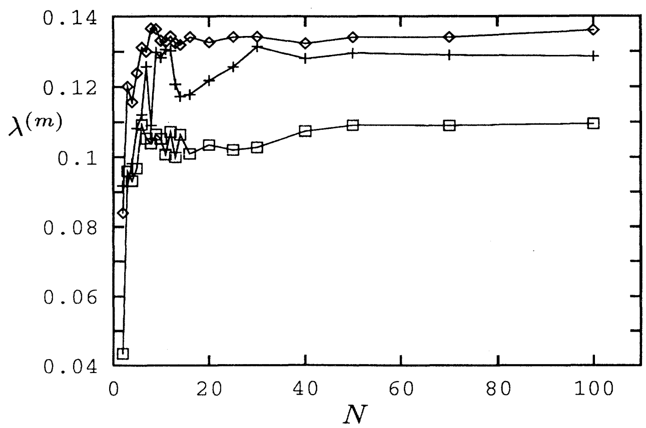

FIG. 4. The largest Liapunov exponent as a function of N for $\nu_c^{-1} = 0.8$ and $\nu = 0.1$ (diamonds), $\nu_c^{-1} = 0.8$ and $\nu = 0.01$ (crosses), $\nu_c^{-1} = 1.5$ and $\nu = 0.1$ (squares). As in Fig. 3, $\lambda^{(m)}$ is calculated for an integration time $\tau = 30\,000N$, with time steps of $\Delta\tau = 0.01$ and perturbations to the position and velocities of the blocks equal to 10^{-5}.

$$\ddot{U}_j = U_{j+1} - 2U_j + U_{j-1} + A\dot{U}_j. \tag{7}$$

Assume that U_j has a solution of the type

$$U_j = u\sin(kaj + \phi)e^{\omega\tau}, \tag{8}$$

where a is the spacing between masses when the elastic forces are in equilibrium. Without losing generality we can take $a = 1$. Substituting this solution into Eq. (7) we find that the eigenvalues of the solution with uniform velocity are given by

$$\omega = \frac{A \pm \sqrt{A^2 - 8(1 - \cos k)}}{2}. \tag{9}$$

In the reference frame that moves with velocity ν, the boundary conditions are expressed as $U_0 = 0$ and $U_{N+1} = U_N$. The first condition gives $\phi = 0$. The second condition implies that $k(N+1) = kN + 2\pi\alpha$ or $(1 + 2\alpha)\pi - k(N+1) = kN$, with α being an integer. The first case is not interesting because U_j becomes independent of j. The second case gives

$$k = \frac{(1 + 2\alpha)\pi}{2N + 1}, \tag{10}$$

with $\alpha = 0, 1, ..., N - 1$, since we must have N normal-mode frequencies. Substituting this solution into Eq. (9), one finds that the eigenvalue with largest magnitude $\omega^{(m)}$ occurs when $\alpha = (N - 1)/2$. It is not difficult to show that $\mathrm{Re}(\omega^{(m)}) > 0$, i.e., the solution is unstable, and

$$|\omega^{(m)}| = \sqrt{2 - 2\cos\frac{N\pi}{2N + 1}}. \tag{11}$$

The largest eigenvalue is a monotonically increasing function of N and converges to 2 as N gets large. Since for this orbit $\lambda^{(m)} = \log_2|\omega^{(m)}|$, we have that $\lambda^{(m)}$ approaches one as N increases. Consequently, the LLE for this orbit has roughly the same qualitative behavior as $\lambda^{(m)}$ shown in Fig. 4. The value of $\lambda^{(m)}$ for the unstable orbit is larger than the numerical asymptotic value of $\lambda^{(m)} \approx 0.1$ found over the chaotic orbit, which suggests the importance of the unstable fixed point in the chaotic dynamics.

We have also numerically studied the second largest Liapunov exponent. Somewhat surprisingly we found it to be identically zero for all N and for representative parameters. This nongeneric result emphasizes the need for analyzing specific systems to obtain the relationship between SOC and chaos.

In summary, we have studied a mechanical system governed by coupled ordinary differential equations that manifests a power law relationship of number of events versus event size, which is taken as the definition of self-organized criticality. We have shown that the critical dynamics of the SOC attractor has the largest Liapunov exponent, which approaches a finite value as the system size N becomes larger. Consequently, the conjecture [2] that the SOC attractor has power law divergence of nearby trajectories is not universally true.

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